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Isomorphic Embeddings of Abstract Interval Systems

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Abstract. We study new abstract algebraic systems generalizing the system of real compact intervals with addition and multiplication by scalar and the isomorphic embedding of these systems into systems having group properties with respect to addition.

1. Introduction

We study the algebraic relations between: i) extended interval arithmetic over normal (proper) intervals using inner (nonstandard) arithmetic operations [3], [4], and, ii) extended interval arithmetic using improper intervals [1], [2], [10]. We have shown in [5] that the first system is a "projection" of the second one on the set of proper intervals. Here we continue our work from [6] aiming to show that the second system is an isomorphic algebraic extension of the first one. We study the essential algebraic properties of the system of intervals necessary for such isomorphic embedding. To this end we deliberately exclude from consideration the inclusion relation between intervals with the corresponding lattice operations involved, concentrating on the properties of the operations addition, subtraction and multiplication by scalar. A similar approach has been used in [9], where more general systems (e.g. convex compact sets in $\mathbb{R}^n$) are studied. An essential difference between our work and [9] is that we are able to construct isomorphic embeddings, whereas the embedding in [9] is not isomorphic. To achieve an isomorphism we first extend the quasilinear space [8] by means of a complete second distributivity law, which involves negation and inner addition [3], [5]. The modified structure thus obtained, called extended quasilinear system, is isomorphically embedded in an analogue of a linear system, called extended linear system, having group properties with respect to addition. Particular systems of intervals which are extended groups and extended linear systems have been considered in [1], [2] and other sources; here we give an abstract algebraic theory of these systems.

In Section 2 we shortly repeat some of the basic concepts from [6], simplifying the definition of extended semigroup (e.g. we do not require here uniqueness

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of the negation). In Section 3 we state in a modified form the basic embedding theorem from [6] and give a complete proof (due to space limit the proof of this theorem has been omitted in [6]). We hope that the inductive exposition of the proof will contribute to a better understanding of the role of the binary and unary operations "addition," "negation," "opposite," and "dual." In Section 4 we consider a semigroup with multiplication by scalar. Using some properties naturally extracted from interval arithmetic, we introduce a so-called extended quasilinear system, which generalizes the interval structure $(I(\mathbb{R}), +, \cdot)$ [3]–[5]. We show that the extended quasilinear system can be isomorphically embedded into an analogue of a linear system, called "extended linear system."

2. Extended Semigroup and Its Isomorphic Embedding

Extended semigroup. An Abelian (commutative) semigroup is a system $(S, +)$, such that $(A + B) + C = A + (B + C)$ and $A + B = B + A$ for A, B, $C \in S$. Our starting point is a class of Abelian semigroups $(S, +)$, satisfying the assumption (T): for every ordered pair (A, B), A, $B \in S$, at least one of the following two statements hold true: i) $A + X = B$ is uniquely solvable, symbolically $A \leq_{\omega} B$, ii) $B + Y = A$ is uniquely solvable, symbolically $B \leq_{\omega} A$ [6]. If both statements i)–ii) hold true simultaneously, we write $A =_{\omega} B$. If $A \leq_{\omega} B$ and $A \neq_{\omega} B$, we write $A <_{\omega} B$. Assumption (T) implies cancellation; indeed $A + D = B + D = E$ and (T) imply that A and B are unique solutions of same equations, hence $A = B$. Therefore the semigroup $(S, +)$ can be embedded in an Abelian group [9]. Recall that an Abelian group is a commutative semigroup $(G, +)$ with: 1) a unique null element 0, such that $a + 0 = a$, for all $a \in G$, and 2) a unique additive inverse (opposite) operator "opp": $G \longrightarrow G$, such that $a + \mathrm{opp}(a) = 0$, for all $a \in G$. In an Abelian group $(G, +)$ for every $a, b \in G$: 1) the equation $a + x = b$ has a unique solution $x = b + \mathrm{opp}(a)$; 2) $a + x = b + x \Longrightarrow a = b$ (cancellation law). We say that a semigroup S is proper, if S is not a group itself.

We impose certain additional assumptions on $(S, +)$, abstracted from interval arithmetic, naturally arriving to the following:

DEFINITION 2.1. A system $(S, S_0, +, \mathrm{neg})$ is called an *extended semigroup (with subgroup* S_0), iff: 1) $(S, +)$ is a proper commutative semigroup satisfying assumption (T); 2) S_0 is a proper subgroup of S ($S_0 \subset S$, $S_0 \neq S$) such that: i) $S_0 \neq \{0\}$, where 0 is the null element of S_0; ii) S_0 is maximal with respect to "$\subset$", that is for any other subgroup S_0' of S we have $S_0' \subset S_0$; 3) there is an operator neg: $S \longrightarrow S$, called negation, such that: i) $\mathrm{neg}(A) = 0 \Longleftrightarrow A = 0$, $A \in S$; ii) $\mathrm{neg}(\mathrm{neg}(A)) = A$, $A \in S$; iii) $\mathrm{neg}(A + B) = \mathrm{neg}(A) + \mathrm{neg}(B)$, A, $B \in S$; iv) $\mathrm{neg}(P) + P = 0$, $P \in S_0$.

Since S_0 is a group, there is an additive inverse (opposite) operator in S_0, denoted by "$-$", so that $(-P) + P = 0$ for $P \in S_0$. Condition iv) can be also written: $\mathrm{neg}(P) = -P$, $P \in S_0$. Properties i)–iv) mean that "neg" is an automorphism in S which isomorphically extends "opposite" from S_0 into S. Note that by definition an extended semigroup and a group both have a binary, a unary and a nullary operation (addition, negation, resp. opposite, and null element), whereas a (usual) semigroup is supposed to have only a binary operation. We shall rename the operator "neg" in S by "$-$" as traditionally done in interval arithmetic. The class of all extended semigroups is denoted by $\mathcal{XS}$. We introduce in $(S, S_0, +, -) \in \mathcal{XS}$ three new binary operations by means of the negation operator as follows. *Subtraction* is defined by $A - B = A + (-B)$, $A, B \in S$. *Inner addition* is defined by:

$$A +^- B = \begin{cases} X|_{(-A)+X=B}, & A \leq_\omega B, \\ Y|_{(-B)+Y=A}, & B \leq_\omega A, \end{cases} \tag{2.1}$$

where $X|_{(-A)+X=B}$ denotes the solution of $(-A)+X = B$. *Inner subtraction* is defined by $A -^- B = A +^- (-B) = \{-X|_{B=A+X}, \text{ if } A \leq_\omega B;\ Y|_{A=B+Y}, \text{ if } B \leq_\omega A\}$. We have $A +^- B = B +^- A$, $-(A +^- B) = (-A) +^- (-B)$, $A +^- (-A) = A -^- A = 0$, $(A +^- B) +^- C = A +^- (B +^- C)$ iff $A \leq_\omega B$ and $C \leq_\omega B$ (see [6]; many properties for intervals [3]–[5] are valid for the abstract case as well).

Factor group with negation. Consider the classical embedding of S into the set $G = S^2 / \sim$ consisting of all pairs (A, B), $A, B \in S$, factorized by the equivalence relation $\sim$: $(A, B) \sim (U, V) \Longleftrightarrow A + V = B + U$, $A, B, U, V \in S$. Addition in G defined by $(A, B) + (C, D) \equiv (A + C, B + D)$ turns G into a group; the opposite element in G is $\text{opp}(A, B) = (B, A)$.

Assumption (T) subdivides G into three disjoint subsets: $G_+ = \{(A, B) \mid B <_\omega A\}$, $G_- = \{(A, B) \mid A <_\omega B\}$, $G_0 = \{(A, B) \mid A =_\omega B\}$, so that $G = G_+ \cup G_0 \cup G_-$. The semigroup $(S, +)$ is isomorphically embedded in $(G, +)$ under $\varphi : S \longrightarrow G_{+,0} = G_+ \cup G_0$ with $\varphi(A) \equiv (A, 0)$, $A \in S$. For $P \in S_0$ we have $\varphi(P) = (P, 0) = (0, -P)$, since $P + (-P) = 0$ (S_0 is a group!). The image of S under φ is $\varphi(S) = G_{+,0}$; moreover, we have $\varphi(S_0) = G_0$ and $\varphi(S_+) = G_+$, where $S_+ = S \setminus S_0$ and G_0 is a subgroup of G. We may use a unique representation of the factorized pairs, by writing the elements of $G_{+,0}$ in the form $(A, 0)$ with $A \in S$, and those of G_- in the form $(0, B)$ with $B \in S_+$. The elements of G_0 are called *degenerate*, those of G_+ are *proper* and those of G_- are *improper*. We shall say that two elements of G are of *same type* if they both belong either to G_+, or to G_-. In the factor group G we introduce neg: $G \longrightarrow G$ by $\text{neg}(A, B) \equiv (-A, -B)$. It is easy to see that "neg" satisfies the properties: N1) $\text{neg}(a) = 0 \Longleftrightarrow a = 0$, $a \in G$; N2) $\text{neg}(\text{neg}(a)) = a$, $a \in G$; N3) $\text{neg}(a + b) = \text{neg}(a) + \text{neg}(b)$, $a, b \in G$; N4) $\text{neg}(p) + p = 0$ for $p \in G_0$; N5) $\text{neg}(a) + a \neq 0$ for $a \in G \setminus G_0$.

Properties N1)–N4) show that "neg" isomorphically extends negation from S into G. Note that both "neg" and "opp" satisfy properties N1)–N4) in G, that is, N1–N4) remain true if we formally replace "opp" for "neg". Both "opp" and "neg" coincide on G_0. However, "neg" and "opp" are distinct on $G \setminus G_0$; indeed, the elements a and $\text{neg}(a)$ for $a \in G \setminus G_0$ are of same type, whereas a and $\text{opp}(a)$ are not. Thus "opp" and "neg" are two distinct isomorphic extensions of the operator "opp" from G_0 into G. Instead of N5) "opp" satisfies $\text{opp}(a) + a = 0$, $a \in G$. The *symmetric* elements $\text{neg}(a) + a$ form a subgroup of G [7].

The operator dual: $G \longrightarrow G$ defined by $\mathrm{dual}(A,B) \equiv \mathrm{neg}(\mathrm{opp}(A,B)) = \mathrm{neg}(B,A) = (-B,-A)$ satisfies: D1) $\mathrm{dual}(a) = 0 \Longleftrightarrow a = 0$, $a \in G$; D2) $\mathrm{dual}(\mathrm{dual}(a)) = a$, $a \in G$; D3) $\mathrm{dual}(a+b) = \mathrm{dual}(a) + \mathrm{dual}(b)$, $a, b \in G$; D4) $\mathrm{dual}(p) = p$, $p \in G_0$; D5) $\mathrm{dual}(a) + a \in G_0$ for $a \in G \setminus G_0$.

Thus "dual" isomorphically extends the "identity" from S into G. Both "dual" and "identity" in G satisfy properties D1)–D4); they coincide on G_0 but for $a \in G \setminus G_0$, the elements a and $\mathrm{dual}(a)$ are not of same type. Note that "identity" does not satisfy D5.

In a factor group G generated by $S \in \mathcal{XS}$ the operator "neg" will be denoted by "$-$" as we do in S. The factor group will be denoted fully by $(G, G_0, +, -)$ to remind the existence of a subgroup G_0 and a negation in G. A factor group $(G, G_0, +, -)$ generated by an extended semigroup $(S, S_0, +, -)$ will be called an *extended factor group*.

A pair of the form $(A; \alpha)$, $A \in S \in \mathcal{XS}$, $\alpha \in \{+, -\}$, is called a *directed element (of S)* [6]. The *direction* α of the directed element $a = (A; \alpha)$ is denoted $\alpha = \tau(a)$; the *projection of a on S* is $\mathrm{pro}(a) = A$. Directed pairs play a role similar to the role of the pairs $(N; \alpha)$, $N \in \mathbb{N}$, $\alpha \in \{+, -\}$, for the definition of integers ($\mathbb{N}$ is the set of natural numbers).

In what follows we find an isomorphism between the factorized and directed pairs.

3. The Isomorphism between Factorized and Directed Pairs

We define D as the set of all pairs of the form $(A; \alpha)$, $A \in S$, $\alpha \in \{+, -\}$, factorized by the equivalence relation: $(A; \alpha) \sim (B; \beta)$, iff $A = B \in S_0$. In other words we shall not distinguish between the pairs $(P; +)$ and $(P; -)$ for $P \in S_0$. We shall briefly call D the *directed set over S*. Define three disjoint sets of directed elements by: $D_+ = \{(A; +) \mid A \in S_+\}$, $D_- = \{(A; -) \mid A \in S_+\}$, $D_0 = \{(P; \alpha) \mid P \in S_0\}$. Denote also $D_{+,0} = \{(A; +) \mid A \in S\}$. We have $D = D_{+,0} \cup D_- = D_+ \cup D_- \cup D_0$.

PROPOSITION 3.1. *Let* $(S, S_0, +, -) \in \mathcal{XS}$ *generate the extended factor group* $G = (G, G_0, +, -)$ *and D be the directed set over S. The mapping* $\psi : G \longrightarrow D$*, defined by* $\psi(A, 0) = (A; +)$*,* $A \in S$*;* $\psi(0, B) = (-B; -)$*,* $B \in S_+$*, is isomorphic. Addition in D under* ψ *is:* $(A; \alpha) + (B; \beta) \equiv (A +^{\alpha\beta} B; \tau)$*, with* $\tau = \{\alpha,\ \text{if } B \le_\omega A; \beta,\ \text{if } A <_\omega B\}$*.*

Proof. To construct an isomorphism ψ between G and D, we first define a bijection $\psi : G_{+,0} \longrightarrow D_{+,0}$ by setting for $(A, 0) \in G_{+,0}$:

$$\psi(A, 0) = (A; +) \in D_{+,0}, \quad A \in S. \tag{3.1}$$

Define addition in $D_{+,0}$ by $(A; +) + (B; +) = (A + B; +)$, then ψ is an isomorphism between the semigroups $(G_{+,0}, +)$ and $(D_{+,0}, +)$. Let us extend the mapping ψ so that it becomes a isomorphic mapping from G onto D. This in particular means that i) $\psi(G_-) = D_-$, that is the image of G_- is D_-, and, ii) ψ is an isomorphism between G_- and D_-. From i) we have $\psi(0, B) \in D_-$, for $B \in S_+$, that is

$$\psi(0, B) = \big(f(B); -\big) \in D_-, \quad B \in S_+, \tag{3.2}$$

with some mapping $f : S_+ \longrightarrow S_+$ and from ii) we have $(f(A + B); -) = (f(A); -) + (f(B); -)$. If the addition rule in D_- is assumed to be in the form $(A; -) + (B; -) = (A + B; -)$, we obtain $f(A) + f(B) = f(A + B)$. Let us assume that the mapping f can be determined on $S = S_+ \cup S_0$ (not only on S_+!) in such a way that ψ, now given by

$$\psi(0, B) = \big(f(B); -\big) \in D_{-,0}, \quad B \in S = S_+ \cup S_0, \tag{3.3}$$

is an isomorphism between $D_{-,0} = D_- \cup D_0$ and $G_{-,0} = G_- \cup G_0$. Since ψ is already defined on D_0 by means of (3.1) we must put (3.3) into agreement with the definition of f on S_0. For $B = P \in S_0$ equality (3.1) produces $\psi(P, 0) = (P; +) \in D_0$ or, equivalently,

$$\psi(0, -P) = (P; -) \in D_0, \quad P \in S_0, \tag{3.4}$$

using that for $P \in S_0$, $(P, 0) = (0, -P) \in G_0$, and $(P; +) = (P; -)$. Equation (3.4) can be also written in the form

$$\psi(0, P) = (-P; -) \in D_0, \quad P \in S_0. \tag{3.5}$$

Comparing (3.5) and (3.3) gives that $f(P) = -P$, for $P \in S_0$. From (3.3) using that ψ is constructed to be an isomorphism we have to ensure $f(A) + f(B) = f(A + B)$, $A, B \in S$. Obviously, if f is a negation on S, then both requirements: i) $f(P) = -P$ on S_0, and, ii) $f(A + B) = f(A) + f(B)$ for $A, B \in S$, which are necessary for ψ to be isomorphic, will be satisfied. Under the assumption $f = \mathrm{neg}$ formula (3.2) becomes

$$\psi(0, B) = (-B; -) \in D_-, \quad B \in S_+. \tag{3.6}$$

Note that the expressions (3.1) and $\psi(0, B) = (-B; -)$, $B \in S$, produce same results for $A, B \in S_0$, hence (3.6) is valid for $B \in S$ as well. The equalities (3.1) and (3.6) define a bijection between G and D. It can be directly checked that ψ preserves the operation "+" if we accept the following definition of addition on D:

$$(A; \alpha) + (B; \beta) \;\equiv\; (A +^{\alpha\beta} B; \tau), \tag{3.7}$$

$$\tau \;\equiv\; \begin{cases} \alpha, & \text{if } B \leq_\omega A, \\ \beta, & \text{if } A <_\omega B. \end{cases} \tag{3.8}$$

The mapping ψ is an isomorphism between G and D, which implies that $(D, +)$ is an extended group. □

Remark. The product $\alpha\beta$ of two binary variables in (3.7) means: $++ = -- = +$, $+- = -+ = -$; also $+^+ = +$ and "$+^-$" is the inner addition defined by (2.1). For example, for $\alpha = \beta = -$, (3.7) reads: $(A; -) + (B; -) = (A + B; -)$. The binary variable τ in (3.8) means the direction of the bigger element $A, B \in S$ (with respect to $\leq_\omega$).

COROLLARY 3.1. *The automorphisms in D under ψ are* $\mathrm{neg}(A;\alpha) = (-A;\alpha)$, $\mathrm{opp}(A;\alpha) = (-A;-\alpha)$ *and* $\mathrm{dual}(A;\alpha) = (A;-\alpha)$.

Proof. Using that $\psi : G \longrightarrow D$ is isomorphic, express $\pi \in \{\mathrm{opp}, \mathrm{dual}, \mathrm{neg}\}$ in D under ψ, by means of $\pi(A;\alpha) = \psi(\pi(\psi^{-1}(A;\alpha)))$. □

According to Proposition 3.1 G and D are isomorphic, therefore the system $(D, D_0, +, -)$, which corresponds to the extended factor group $(G, G_0, +, -)$, is also a group; we shall call it *the extended directed group (generated by $S \in \mathcal{XS}$)*.

The class of all extended groups (without regard to being factor or directed) will be denoted by $\mathcal{XG}$; we shall consider G and D as two copies of one and the same algebraic system from the class $\mathcal{XG}$. We shall further use notation G only to mean the extended *factor* group and notation D to mean either the *factor* or the *directed* group. Note that for the algebraic completion of the extended semigroup up to an extended group we substantially make use of the negation operator in the original extended semigroup. A typical formula is: $\mathrm{opp}(A;\alpha) = (-A;-\alpha)$, showing that the opposite elements in D, cannot be expressed without negation. On the other side the automorphism $\mathrm{dual}(A, B) = (-B, -A)$ cannot be expressed in G without negation, too.

The relation between “neg”, “opp”, and “dual” in $D \in \mathcal{XG}$ obtains the form: $-a = \mathrm{opp}(\mathrm{dual}(a)) = \mathrm{dual}(\mathrm{opp}(a))$, $a \in D$. For “dual” we shall use the brief notation: $\mathrm{dual}(a) = a_-$. For uniformity we set $a_+ = a$, $a \in D$; then we can use the convenient notation a_λ, where $\lambda \in \{+, -\}$ is a binary variable. Note that the equations $a_\lambda = b$ and $a = b_\lambda$ are equivalent for $a \in D$. The relation between the three automorphic mappings leads to a useful symbolic notation for the additive inverse operation: $\mathrm{opp}(a) = -(a_-) = (-a)_- = -a_-$, $a \in D$. In computations we may refrain from using “neg(a)”, “dual(a)”, and “opp(a)” in favour of $-a$, a_- and $-a_-$, resp. For $a \in D$ we have $a + a \equiv a - a + a + a_-$.

4. Isomorphic Embedding of a Quasilinear System

We introduce multiplication by scalars in the systems S, D and G. Let $(S, S_0, +)$ satisfy assumptions 1) and 2) of Definition 2.1 (there is no need to assume 3), since negation will be obtained “free” from multiplication by -1). Define multiplication by scalar “$\cdot$” (the dot may be omitted) in the cartesian product of $\mathbb{R}$ and S (instead of $\mathbb{R}$ we can take $\mathbb{C}$ or some other field) as follows:

DEFINITION 4.1. An operation defined for every pair (p, A), with $p \in \mathbb{R}$, $A \in S$, denoted by $p \cdot A \equiv pA$, is called *multiplication by scalar*, if for $A, B \in S$, $p, q \in \mathbb{R}$ the following properties i)–v) are satisfied:

i) associativity of multiplication by scalar: $p(qA) = (pq)A$;

ii) first distributive law: $p(A + B) = pA + pB$;

iii) second distributivity law:

$$(p + q)A = pA +^{\sigma(p)\sigma(q)} qA, \tag{4.1}$$

where $\sigma(p) = \{+, \text{ if } p \geq 0; \ -, \text{ if } p < 0\}$ and "$+^-$" is defined by (2.1) assuming that the symbol $-A$ in (2.1) is defined by $-A \equiv (-1)A$;

iv) $1 \cdot A = A$; and

v) $(-1) \cdot P + P = 0$, for $P \in S_0$.

It is easily seen that $p(A+^- B) = pA+^- pB$. It can also be verified that $-A \equiv (-1)A$ is negation in the sense of Definition 2.1, hence $(S, S_0, +, -) \in \mathcal{XS}$. Note that setting $p = 1$, $q = -1$ in (4.1), implies $A +^- (-A) = A -^- A = 0$, as expected. The extended semigroup $(S, S_0, +, -)$ endowed with multiplication by scalar will be denoted $S(\mathbb{R}) = (S, S_0, +, \mathbb{R}, \cdot)$ and called *extended quasilinear system (over* $\mathbb{R}$). We shall denote by $\mathcal{XQ}$ the class of all extended quasilinear systems. According to Proposition 3.1 an extended semigroup can be isomorphically embedded in a extended group. We shall next extend the multiplication by scalar from $S(\mathbb{R}) \in \mathcal{XQ}$ into D, resp. G, in order to embed isomorphically any extended quasilinear system into a system having group properties with respect to addition.

PROPOSITION 4.1. *Let* $S(\mathbb{R}) = (S, S_0, +, \mathbb{R}, \cdot) \in \mathcal{XQ}$ *and let* $D \in \mathcal{XG}$ *be an extended group generated by* S*. Assume* $A, B \in S$*,* $p, q \in \mathbb{R}$*,* $a, b \in D$ *(and, in particular,* $a, b \in G$*). Then:*

i) the setting $p(A, B) \equiv (pA, pB)$ *isomorphically extends the multiplication by scalar in* G*; multiplication by "*-1*" is negation in* G*;*

ii) $p(A; \alpha) \equiv (pA; \alpha)$ *isomorphically extends the multiplication by scalar from* S *to* D*; multiplication by "*-1*" is negation in* D*;*

iii) $p(qa) = (pq)a$*,* $p(a + b) = pa + pb$*,* $1a = a$ *and* $(-1)a + a = 0$ *for* $a \in D_0$*;*

iv) the following second distributivity law holds true:

$$(p + q)a_{\sigma(p+q)} = pa_{\sigma(p)} + qa_{\sigma(q)}, \quad a \in D, \ p, q \in \mathbb{R}, \tag{4.2}$$

which can be also written in the form: $(p + q)a = pa_\lambda + qa_\mu$*, with* $\lambda = \sigma(p)\sigma(p + q)$*,* $\mu = \sigma(q)\sigma(p + q)$*;*

v) $(-1)a + a = 0$ *for* $a \in D_0$*.*

PROPOSITION 4.2. *For* $p, q \in \mathbb{R}$*,* $\lambda \in \{+, -\}$*,* $d \in D$*,* $s \equiv p^2 - q^2 \neq 0$*, the equation* $px + qx_\lambda = d$ *has a unique solution* $x = s^{-1}(pd - qd_{-\lambda})_{\sigma(s)}$.

If $\lambda = \sigma(p / q)$ using (4.2) we have: $x = (p + q)^{-1} d_{\sigma(1+q/p)}$.

We shall denote an extended factor group endowed with multiplication by scalar by $(G, G_0, +, \mathbb{R}, \cdot)$ omitting the "$-$" as special case of multiplication by scalar. Similarly we denote the respective extended directed group by $(D, D_0, +, \mathbb{R}, \cdot)$; we have $(G, G_0, +, \mathbb{R}, \cdot) \cong (D, D_0, +, \mathbb{R}, \cdot)$ under the isomorphism ψ from Proposition 3.1.

The algebraic system $(D, D_0, +, \mathbb{R}, \cdot)$ is called *extended linear system (over* $\mathbb{R}$); the class of all extended linear systems is denoted by $\mathcal{XL}$. Using this terminology we can state Proposition 4.1 as: every extended quasilinear system can be isomorphically embedded in an extended linear system.

EXAMPLE. Let us illustrate the above results to the set $I(\mathbb{R})$ of all compact intervals on $\mathbb{R}$. Denote $A = [a^-, a^+] \in I(\mathbb{R})$. Let "+" be the interval addition: $A + B = \{x + y \mid x \in A, y \in B\}$ and "$-$" be the operator negation: $-A = \{-x \mid x \in A\}$; end-pointwise: $A + B \equiv [a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+]$, resp., $-[a^-, a^+] = [-a^+, -a^-]$. The interval arithmetic system $(I(\mathbb{R}), +, -)$ is an extended semigroup; it satisfies assumption (T), with $A \leq_\omega B$ meaning $a^+ - a^- \leq b^+ - b^-$. The inner addition is $[a^-, a^+] +^- [b^-, b^+] = \{[a^- + b^+,\ a^+ + b^-],$ if $B \leq_\omega A$; $[a^+ + b^-,\ a^- + b^+],$ if $A <_\omega B\}$. The multiplication by scalar in $I(\mathbb{R})$, defined by $pA \equiv \{px \mid x \in A\} = \{[pa^-, pa^+],$ if $p \geq 0$; $[pa^+, pa^-],$ if $p < 0\}$, satisfies the second distributivity law in the form (4.1), hence $(I(\mathbb{R}), +, \cdot)$ is an extended quasilinear system. The latter possesses a rich algebraic structure involving inner operations [3]–[6]. In the case of $I(\mathbb{R})$ our embedding theorems state: 1) every extended semigroup $(I(\mathbb{R}), +, -)$ is isomorphically embedded into an extended group $(D, +, -) \in \mathcal{XG}$, where D is the set of *extended* (also called "directed" or "generalized") intervals of the form $[a^-, a^+]$, $a^-, a^+ \in \mathbb{R}$ (no restriction $a^- \leq a^+$ assumed); 2) every extended quasilinear system $(I(\mathbb{R}), +, \cdot)$ is isomorphically embedded into an extended linear system of directed intervals which is a group with respect to "+" and possesses a second distributive law of the form (4.2). The operators "dual" and "opp" in the extended (group/linear) interval system are: dual $[a^-, a^+] = [a^+, a^-]$, opp$[a^-, a^+] = [-a^-, -a^+]$ (cf. [1], [2]). We can also work with extended intervals in *directed form*: $[A; \alpha]$, $A \in I(\mathbb{R})$, $\alpha \in \{+, -\}$ [5].

5. Conclusion

Our work reveals the role of the extended algebraic systems (semigroups, groups, quasilinear and linear systems) in the algebraic study of interval sets, in particular, it points out the role of the inner operations, e.g. (2.1), and the complete second distributivity law (4.1); for intervals the latter has been first reporteded in [3]. From our work it becomes obvious that computations in an extended linear system $(D, +, \cdot)$ are similar to those in a familiar linear system up to the following two main differences: i) in D we have four automorphisms ("neg", "opp", "dual", and "identity") against two ("opp" and "identity") in an usual linear system, and ii) a slightly modified second linear law involving the operator "dual" takes place in D.

Future work can be done in the following directions. First, extended quasilinear and extended linear systems of n-dimensional elements can be introduced in the form $(S^n, +, \cdot)$, resp. $(D^n, +, \cdot)$, wherein "+" and "$\cdot$" are defined componentwise. Second, using multiplicative notations we can derive multiplicative analogues to the additive semigroup and group systems (as noted in [6]). The link between addition and multiplication by scalars in $\mathcal{XQ}$, resp. $\mathcal{XL}$, can be extended towards a link between additive and multiplicative operations. To this end one can use an abstraction of the corresponding distributive relations in $(I(\mathbb{R}), +, \times)$, resp. $(D, +, \times)$ (cf. [5], [6]) to study abstract systems which are close to rings and algebras [7]. Third, we may consider the extension of a metric, norm, and inclusion from S to D; such extensions have been already constructed for spaces of factorized pairs [2], [9] and can be easily adopted for spaces of directed elements. Thus the analysis of interval functions developed in [4] can be extended in the spirit of this work for functions which values are directed elements, and, in particular, are directed intervals. Fourth, this work outlines a new approach to the exposition of the theory of complex intervals, and many results can be transferred for compact convex sets as well. In particular, the complete second distributivity law (4.1) holds true for compact convex sets.

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